

An operator-theoretical treatment of the Maskawa-Nakajima equation in the massless abelian gluon model

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Abstract

The Maskawa-Nakajima equation has attracted considerable interest in elementary particle physics. From the viewpoint of operator theory, we study the Maskawa-Nakajima equation in the massless abelian gluon model. On the basis of the Schauder fixed-point theorem, we first show that there is a nonzero solution to the Maskawa-Nakajima equation when the parameter λ satisfies $\lambda > 2$. Moreover, we show that the solution is infinitely differentiable and strictly decreasing. We thus conclude that the massless abelian gluon model exhibits the spontaneous chiral symmetry breaking when $\lambda > 2$. On the basis of the Banach fixed-point theorem, we next show that there is a unique solution 0 to the Maskawa-Nakajima equation when $0 < \lambda < 1$, from which we conclude that the model realizes the chiral symmetry when $0 < \lambda < 1$.

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1 Introduction and preliminaries

The Maskawa-Nakajima equation has attracted considerable interest in elementary particle physics. The equation is applied to many models such as a massive abelian gluon model [10, 11, 9], a massless abelian gluon model [5], a QCD(quantum chromodynamics)-like model [8], a technicolor model [17] and a top quark condensation model [12].

The Maskawa-Nakajima equation in the massless abelian gluon model is the following nonlinear integral equation [10, 11, 5, 9] :

$$(1.1) \quad u(x) = \frac{\lambda}{2} \int_{\varepsilon}^{\Lambda} \frac{1}{y+x+|y-x|} \frac{y u(y)}{y+u(y)^2} dy, \quad \varepsilon \leq x \leq \Lambda,$$

where the infrared cutoff $\varepsilon > 0$ is small enough, while the ultraviolet cutoff $\Lambda > 0$ is large enough. Here, $\lambda = \frac{3a^2}{4\pi^2} > 0$ is a parameter with $a > 0$ called the gauge coupling constant.

The adopted gauge group of the massless abelian gluon model is $U(1)$, an abelian group, and the model describes the gauge interactions among quarks and massless gluons.

The solution u to the Maskawa-Nakajima equation (1.1) is the mass function of the quark, and it is known that the quark mass breaks the symmetry called the chiral symmetry of the model. So, if there is a nonzero solution to the Maskawa-Nakajima equation (1.1), then the massless abelian gluon model is said to exhibit the spontaneous chiral symmetry breaking; if there is a unique solution 0 to the Maskawa-Nakajima equation (1.1), then the massless abelian gluon model is said to realize the chiral symmetry. The reason why Professor Maskawa reconsider the spontaneous chiral symmetry breaking in a renormalizable model of strong interaction and so on is mentioned in his Nobel lecture.

The present author thinks that the Maskawa-Nakajima equation plays a role similar to that of the BCS gap equation in the BCS model [1, 3] for superconductivity. If there is a nonzero solution to the BCS gap equation, then the BCS model exhibits the spontaneous breaking of the $U(1)$ symmetry; if there is a unique solution 0 to the BCS gap equation, then the BCS model realizes the $U(1)$ symmetry. The existence and uniqueness of the solution to the BCS gap equation as well as its properties are studied in [13, 2, 14, 4, 6, 7, 15, 16].

Let $w \in C[\varepsilon, \Lambda]$ be

$$(1.2) \quad w(x) = \frac{4\varepsilon}{\lambda} \sqrt{\frac{\varepsilon}{\Lambda x}}, \quad \varepsilon \leq x \leq \Lambda.$$

We consider the following subset of the Banach space $C[\varepsilon, \Lambda]$:

$$(1.3) \quad V = \left\{ u \in C[\varepsilon, \Lambda] : w(x) \leq u(x) \leq \frac{\lambda}{4} \sqrt{\Lambda} \text{ at all } x \in [\varepsilon, \Lambda] \right\}.$$

For $u \in V$, we define a nonlinear integral operator A by

$$(1.4) \quad Au(x) = \frac{\lambda}{2} \int_{\varepsilon}^{\Lambda} \frac{1}{y + x + |y - x|} \frac{y u(y)}{y + u(y)^2} dy, \quad \varepsilon \leq x \leq \Lambda.$$

Then $Au(x)$ is well-defined at every $x \in [\varepsilon, \Lambda]$. Note that $Au(x)$ coincides with the right side of (1.1). So we look for a fixed point of the nonlinear integral operator A . Note also that $Au(x)$ is rewritten as

$$(1.5) \quad Au(x) = \frac{\lambda}{4} \left\{ \frac{1}{x} \int_{\varepsilon}^x \frac{y u(y)}{y + u(y)^2} dy + \int_x^{\Lambda} \frac{u(y)}{y + u(y)^2} dy \right\}, \quad \varepsilon \leq x \leq \Lambda.$$

Assume that the function u is a fixed point of the operator A and that u is smooth on $[\varepsilon, \Lambda]$. Then, by (1.5),

$$(1.6) \quad x^2 u''(x) + 2xu'(x) + \frac{\lambda}{4} \frac{xu(x)}{x + u(x)^2} = 0, \quad \varepsilon \leq x \leq \Lambda.$$

Assume also that $u(x) > 0$ is small enough at x large enough. It then follows from (1.6) that at x large enough,

$$(1.7) \quad x^2 u''(x) + 2xu'(x) + \frac{\lambda}{4} u(x) = 0.$$

See Kugo and Nakajima [9]. Studying the solution to the ordinary differential equation (1.7) leads to the conclusions that the massless abelian gluon model exhibits the spontaneous chiral symmetry breaking for $\lambda > 1$ and that the model realizes the chiral symmetry

for $0 < \lambda < 1$ (see [5, 9]). Note that (1.7) is obtained under the assumption that $u(x) > 0$ is small enough at x large enough. But one does not know whether or not $u(x) > 0$ is small enough at x large enough. It may happen that $u(x)$ is very large at x large enough. So we address this problem from the viewpoint of operator theory without the assumption that $u(x) > 0$ is small enough at x large enough.

On the other hand, replacing $u(x)$ of (1.1) by $u(x) = \frac{\psi(x)}{\sqrt{x+\varepsilon}}$ and letting Λ tend to infinity change (1.1) into

$$(1.8) \quad \psi(x) = \frac{\lambda}{2} \sqrt{x+\varepsilon} \int_{\varepsilon}^{\infty} \frac{1}{y+x+|y-x|} \frac{y}{\sqrt{y+\varepsilon}} \frac{\psi(y)}{y + \frac{\psi(y)^2}{y+\varepsilon}} dy, \quad \varepsilon \leq x < \infty.$$

Remark 1.1. Because of our mathematical interest, we let $\Lambda \rightarrow \infty$ in (1.8), and so we consider the Banach space $B^0[\varepsilon, \infty)$ in (1.9) below. We can similarly deal with the case where Λ remains finite. See Remark 2.4 below.

We consider the Banach space $B^0[\varepsilon, \infty)$ consisting of all the bounded and continuous functions on $[\varepsilon, \infty)$, and deal with the following subset:

$$(1.9) \quad W = \left\{ \psi \in B^0[\varepsilon, \infty) : \psi(x) \geq 0 \text{ at all } x \in [\varepsilon, \infty) \right\}.$$

For $\psi \in W$, we define another nonlinear integral operator B by

$$(1.10) \quad B\psi(x) = \frac{\lambda}{2} \sqrt{x+\varepsilon} \int_{\varepsilon}^{\infty} \frac{1}{y+x+|y-x|} \frac{y}{\sqrt{y+\varepsilon}} \frac{\psi(y)}{y + \frac{\psi(y)^2}{y+\varepsilon}} dy, \quad \varepsilon \leq x < \infty.$$

Then $B\psi(x)$ is well-defined at every $x \in [\varepsilon, \infty)$. Note that $B\psi(x)$ coincides with the right side of (1.8). So we again look for a fixed point of the nonlinear integral operator B .

The paper proceeds as follows. In section 2 we state our main results without proof. In sections 3 and 4 we prove our main results.

2 Main results

We first deal with the case where $\lambda > 2$. For $\lambda > 2$, let ε and Λ satisfy

$$(2.1) \quad \frac{\varepsilon}{\Lambda} \leq \min \left(\frac{1}{16}, \left(\frac{\sqrt{\lambda^2 + 128(\lambda - 2)} - \lambda}{64} \right)^2 \right).$$

Remark 2.1. As mentioned above, $\varepsilon > 0$ is small enough, while $\Lambda > 0$ is large enough. So (2.1) is clearly fulfilled for $\lambda > 2$.

Theorem 2.2. Assume $\lambda > 2$. Let ε and Λ satisfy (2.1). Let A be as in (1.4).

(a) The nonlinear integral operator $A : V \rightarrow V$ has at least one fixed point $u_0 \in V$. Consequently, u_0 is continuous on $[\varepsilon, \Lambda]$ and $(0 <) w(x) \leq u_0(x) \leq \lambda\sqrt{\Lambda}/4$ at all $x \in [\varepsilon, \Lambda]$. Hence the massless abelian gluon model exhibits the spontaneous chiral symmetry breaking. Moreover, each fixed point $u_0 \in V$ is strictly decreasing on $[\varepsilon, \Lambda]$, and satisfies

$$u_0 \in C^\infty[\varepsilon, \Lambda], \quad u'_0(\varepsilon) = 0.$$

(b) Let $u_0 \in V$ be a fixed point of the nonlinear integral operator $A : V \rightarrow V$ above. If $u_0(x) \leq \sqrt{x}$ at all $x \in [\varepsilon, \Lambda]$, then $A : V \rightarrow V$ has a unique fixed point $u_0 \in V$.

We then deal with the case where $0 < \lambda < 1$.

Theorem 2.3. Assume $0 < \lambda < 1$. Let B be as in (1.10). Then the nonlinear integral operator $B : W \rightarrow W$ has a unique fixed point $0 \in W$. Consequently, the massless abelian gluon model realizes the chiral symmetry.

Remark 2.4. Because of our mathematical interest, we let $\Lambda \rightarrow \infty$ in (1.8) and we consider the Banach space $B^0[\varepsilon, \infty)$ in (1.9), as mentioned in Remark 1.1. Indeed, when $0 < \lambda < 1$, we can similarly deal with the case where Λ remains finite. In this case, we define the operator B on the set

$$W_1 = \{\psi \in C[\varepsilon, \Lambda] : \psi(x) \geq 0 \text{ at all } x \in [\varepsilon, \Lambda]\}$$

instead of the set W (see (1.9)). We can similarly show that the nonlinear integral operator $B : W_1 \rightarrow W_1$ has a unique fixed point $0 \in W_1$, and hence we conclude that the massless abelian gluon model realizes the chiral symmetry when $0 < \lambda < 1$.

3 Proof of Theorem 2.2

In this section we prove Theorem 2.2 in a sequence of lemmas.

We fix $y \in [\varepsilon, \Lambda]$ and study some properties of the function $g : [0, \infty) \rightarrow [0, \infty)$ given by

$$g(X) = \frac{X}{y + X^2}, \quad X \geq 0.$$

Here, X corresponds to $u(y)$ of (1.3) and $g(X)$ to $\frac{u(y)}{y + u(y)^2}$ of (1.4). So X satisfies $w(y) \leq X \leq \lambda\sqrt{\Lambda}/4$ at each fixed $y \in [\varepsilon, \Lambda]$.

Note that (2.1) implies $w(\varepsilon) = \frac{4\varepsilon}{\lambda\sqrt{\Lambda}} < \sqrt{\varepsilon}$.

We first suppose $2 < \lambda < 4$. Since $\lambda^2\Lambda/16 < \Lambda$, then (2.1) implies that at each fixed $y \in [\varepsilon, \lambda^2\Lambda/16]$,

$$w(y) \leq \frac{4y}{\lambda\sqrt{\Lambda}} \leq \sqrt{y} \leq \frac{\lambda}{4}\sqrt{\Lambda}$$

and that at each fixed $y \in [\lambda^2\Lambda/16, \Lambda]$,

$$w(y) \leq \frac{\lambda}{4}\sqrt{\Lambda} \left(\leq \sqrt{y} \leq \frac{4y}{\lambda\sqrt{\Lambda}} \right).$$

A straightforward calculation gives the following tables: For each fixed $y \in [\varepsilon, \lambda^2\Lambda/16]$,

X	$w(y)$		$\frac{4y}{\lambda\sqrt{\Lambda}}$		\sqrt{y}		$\frac{\lambda}{4}\sqrt{\Lambda}$
$g'(X)$		+		+	0	−	
$g(X)$		\nearrow	$\frac{\frac{\lambda}{4}\sqrt{\Lambda}}{y + \frac{\lambda^2\Lambda}{16}}$	\nearrow	$\frac{1}{2\sqrt{y}}$	\searrow	$\frac{\frac{\lambda}{4}\sqrt{\Lambda}}{y + \frac{\lambda^2\Lambda}{16}}$

and for each fixed $y \in [\lambda^2\Lambda/16, \Lambda]$,

X	$w(y)$		$\frac{\lambda}{4}\sqrt{\Lambda}$		\sqrt{y}		$\frac{4y}{\lambda\sqrt{\Lambda}}$
$g'(X)$		+		+	0	−	
$g(X)$		\nearrow	$\frac{\frac{\lambda}{4}\sqrt{\Lambda}}{y + \frac{\lambda^2\Lambda}{16}}$	\nearrow	$\frac{1}{2\sqrt{y}}$	\searrow	$\frac{\frac{\lambda}{4}\sqrt{\Lambda}}{y + \frac{\lambda^2\Lambda}{16}}$

Note that $g\left(\frac{4y}{\lambda\sqrt{\Lambda}}\right) = g\left(\frac{\lambda}{4}\sqrt{\Lambda}\right)$ and that X does not satisfy $X > \frac{\lambda}{4}\sqrt{\Lambda}$. The symbol \nearrow (resp. \searrow) shows that the function g is strictly increasing (resp. strictly decreasing) on each interval.

We next suppose $\lambda \geq 4$. Then (2.1) implies that at each fixed $y \in [\varepsilon, \Lambda]$,

$$w(y) \leq \frac{4y}{\lambda\sqrt{\Lambda}} \leq \sqrt{y} \leq \frac{\lambda}{4}\sqrt{\Lambda}.$$

A straightforward calculation gives the following table: For each fixed $y \in [\varepsilon, \Lambda]$,

X	$w(y)$		$\frac{4y}{\lambda\sqrt{\Lambda}}$		\sqrt{y}		$\frac{\lambda}{4}\sqrt{\Lambda}$
$g'(X)$		+		+	0	−	
$g(X)$		\nearrow	$\frac{\frac{\lambda}{4}\sqrt{\Lambda}}{y + \frac{\lambda^2\Lambda}{16}}$	\nearrow	$\frac{1}{2\sqrt{y}}$	\searrow	$\frac{\frac{\lambda}{4}\sqrt{\Lambda}}{y + \frac{\lambda^2\Lambda}{16}}$

Note again that $g\left(\frac{4y}{\lambda\sqrt{\Lambda}}\right) = g\left(\frac{\lambda}{4}\sqrt{\Lambda}\right)$.

On the basis of these observations above, we now study some properties of the operator A given by (1.4).

Lemma 3.1. *Assume $\lambda > 2$ and assume (2.1) holds. Let w be as in (1.2) and (1.3). Then $Aw(x) > w(x)$ at all $x \in [\varepsilon, \Lambda]$.*

Proof. Since $y^2 + \frac{16\varepsilon^3}{\lambda^2\Lambda} < (1 + \frac{16\varepsilon}{\Lambda})y^2$ and $\sqrt{\frac{\varepsilon}{x}} + \sqrt{\frac{x}{\Lambda}} \leq 1 + \sqrt{\frac{\varepsilon}{\Lambda}}$, it follows from (1.5) that

$$\begin{aligned} Aw(x) &= w(x) \frac{\lambda}{4} \left\{ \frac{1}{\sqrt{x}} \int_{\varepsilon}^x \frac{y^{3/2}}{y^2 + \frac{16\varepsilon^3}{\lambda^2\Lambda}} dy + \sqrt{x} \int_x^{\Lambda} \frac{y^{1/2}}{y^2 + \frac{16\varepsilon^3}{\lambda^2\Lambda}} dy \right\} \\ &> w(x) \frac{\lambda}{1 + \frac{16\varepsilon}{\Lambda}} \left\{ 1 - \frac{1}{2} \left(\sqrt{\frac{\varepsilon}{x}} + \sqrt{\frac{x}{\Lambda}} \right) \right\} \\ &\geq w(x) \frac{\lambda(1 - \sqrt{\frac{\varepsilon}{\Lambda}})}{2(1 + \frac{16\varepsilon}{\Lambda})}. \end{aligned}$$

The inequality $\frac{\lambda(1 - \sqrt{\frac{\varepsilon}{\Lambda}})}{2(1 + \frac{16\varepsilon}{\Lambda})} \geq 1$ follows from (2.1). □

Lemma 3.2. *Assume $\lambda > 2$ and assume (2.1) holds. Let w be as in (1.2) and (1.3), and let $u \in V$. Then $\frac{u(x)}{x + u(x)^2} \geq \frac{w(x)}{x + w(x)^2}$, and hence $Au(x) \geq Aw(x)$ at all $x \in [\varepsilon, \Lambda]$.*

Proof. Let $u \in V$. Then $w(y) \leq u(y) \leq \frac{\lambda}{4}\sqrt{\Lambda}$ at all $y \in [\varepsilon, \Lambda]$ by the definition (1.3). The tables above therefore implies at each fixed $y \in [\varepsilon, \Lambda]$,

$$\frac{u(y)}{y + u(y)^2} \geq \frac{w(y)}{y + w(y)^2}.$$

Hence, at all $x \in [\varepsilon, \Lambda]$,

$$Au(x) \geq \frac{\lambda}{2} \int_{\varepsilon}^{\Lambda} \frac{1}{y + x + |y - x|} \frac{y w(y)}{y + w(y)^2} dy = Aw(x).$$

□

The two lemmas just above are our key lemmas in this section.

Lemma 3.3. *Assume $\lambda > 2$ and assume (2.1) holds. If $u \in V$, then $Au(x) \leq \frac{\lambda}{4}\sqrt{\Lambda}$ at all $x \in [\varepsilon, \Lambda]$.*

Proof. By (1.5), $Au(x) \leq \frac{\lambda}{8} \left\{ \frac{1}{x} \int_{\varepsilon}^x \sqrt{y} dy + \int_x^{\Lambda} \frac{1}{\sqrt{y}} dy \right\} < \frac{\lambda}{4}\sqrt{\Lambda}$. □

Lemma 3.4. *Assume $\lambda > 2$ and assume (2.1) holds. If $u \in V$, then $Au \in C[\varepsilon, \Lambda]$.*

Proof. Let $\varepsilon \leq x_0 \leq \Lambda$. Then

$$|Au(x) - Au(x_0)| \leq \frac{\lambda}{2} \int_{\varepsilon}^{\Lambda} \left| \frac{1}{y + x + |y - x|} - \frac{1}{y + x_0 + |y - x_0|} \right| \frac{\sqrt{y}}{2} dy.$$

The function $(x, y) \mapsto \frac{1}{y + x + |y - x|}$ is uniformly continuous on $[\varepsilon, \Lambda]^2$. Hence, for an arbitrary $\varepsilon_1 > 0$, there is a $\delta > 0$ satifying

$$\left| \frac{1}{y + x + |y - x|} - \frac{1}{y + x_0 + |y - x_0|} \right| < \varepsilon_1, \quad |x - x_0| < \delta.$$

Note that δ depends neither on x , nor on x_0 , nor on y , nor on u . Therefore,

$$|Au(x) - Au(x_0)| < \frac{\lambda \Lambda^{3/2}}{6} \varepsilon_1, \quad |x - x_0| < \delta.$$

□

These four lemmas imply that the set $AV = \{Au : u \in V\}$ is a subset of V .

Lemma 3.5. *Assume $\lambda > 2$ and assume (2.1) holds. Then $AV \subset V$.*

Lemma 3.6. *Assume $\lambda > 2$ and assume (2.1) holds. Then the set AV is relatively compact.*

Proof. By Lemma 3.3, the set AV is uniformly bounded. As mentioned in the proof of Lemma 3.4, the δ does not depend on $u \in V$. Hence the set AV is equicontinuous. The result thus follows from the Ascoli–Arzelà theorem. □

Lemma 3.7. *Assume $\lambda > 2$ and assume (2.1) holds. Then the operator $A : V \rightarrow V$ is continuous.*

Proof. Let $u, v \in V$. Then, at every $x \in [\varepsilon, \Lambda]$,

$$|Au(x) - Av(x)| \leq \frac{\lambda}{2} \|u - v\|_{C[\varepsilon, \Lambda]} \int_{\varepsilon}^{\Lambda} \frac{1}{y + x + |y - x|} dy.$$

Here, $\|\cdot\|_{C[\varepsilon, \Lambda]}$ denotes the norm of the Banach space $C[\varepsilon, \Lambda]$. Since

$$\int_{\varepsilon}^{\Lambda} \frac{1}{y + x + |y - x|} dy \leq \frac{1}{2} \left(1 + \ln \frac{\Lambda}{\varepsilon} \right),$$

it follows $\|Au - Av\|_{C[\varepsilon, \Lambda]} \leq \frac{\lambda}{4} \left(1 + \ln \frac{\Lambda}{\varepsilon} \right) \|u - v\|_{C[\varepsilon, \Lambda]}$. \square

The set V is clearly bounded, closed and convex. The Schauder fixed-point theorem (see e.g. Zeidler [18, p. 61]) thus implies the following.

Lemma 3.8. *Assume $\lambda > 2$ and assume (2.1) holds. The operator $A : V \rightarrow V$ has at least one fixed point $u_0 \in V$. Consequently, $u_0 \neq 0$, and hence the massless abelian gluon model exhibits the spontaneous chiral symmetry breaking.*

Lemma 3.9. *Assume $\lambda > 2$ and assume (2.1) holds. Let $u_0 \in V$ be a fixed point of the operator $A : V \rightarrow V$ given by Lemma 3.8. Then $u_0 \in C^\infty[\varepsilon, \Lambda]$, $u'_0(\varepsilon) = 0$ and u_0 is strictly decreasing on $[\varepsilon, \Lambda]$.*

Proof. Each fixed point $u_0 \in V$ satisfies (1.1), i.e.,

$$\begin{aligned} u_0(x) &= \frac{\lambda}{2} \int_{\varepsilon}^{\Lambda} \frac{1}{y + x + |y - x|} \frac{y u_0(y)}{y + u_0(y)^2} dy \\ &= \frac{\lambda}{4} \left\{ \frac{1}{x} \int_{\varepsilon}^x \frac{y u_0(y)}{y + u_0(y)^2} dy + \int_x^{\Lambda} \frac{u_0(y)}{y + u_0(y)^2} dy \right\}, \quad \varepsilon \leq x \leq \Lambda. \end{aligned}$$

Note that $u_0 \in V$ is continuous and $u_0(x) > 0$ at each $x \in [\varepsilon, \Lambda]$. We then regard the integrals above as the Riemann integrals. Then u_0 is differentiable on $[\varepsilon, \Lambda]$:

$$u'_0(x) = -\frac{\lambda}{4x^2} \int_{\varepsilon}^x \frac{y u_0(y)}{y + u_0(y)^2} dy \leq 0, \quad \varepsilon \leq x \leq \Lambda.$$

Note that the equality $u'_0(x) = 0$ holds at $x = \varepsilon$ only. We thus see that $u'_0(\varepsilon) = 0$, that u_0 is strictly decreasing on $[\varepsilon, \Lambda]$, and that u_0 is infinitely differentiable on $[\varepsilon, \Lambda]$. \square

Lemma 3.10. *Assume $\lambda > 2$ and assume (2.1) holds. Let $u_0 \in V$ be a fixed point of the operator $A : V \rightarrow V$ given by Lemma 3.8. If $u_0(x) \leq \sqrt{x}$ at all $x \in [\varepsilon, \Lambda]$, then $A : V \rightarrow V$ has a unique fixed point $u_0 \in V$.*

Proof. Let $v_0 \in V$ be another fixed point of $A : V \rightarrow V$. Then there are a number t ($0 < t < 1$) and a point $x_0 \in [\varepsilon, \Lambda]$ such that

$$(3.1) \quad u_0(x) \geq t v_0(x) \quad (x \in [\varepsilon, \Lambda]) \quad \text{and} \quad u_0(x_0) = t v_0(x_0).$$

Note that $t v_0(x) \leq u_0(x) \leq \sqrt{x}$ at each fixed $x \in [\varepsilon, \Lambda]$. The three tables above in this section therefore imply

$$\begin{aligned} u_0(x_0) &= \frac{\lambda}{2} \int_{\varepsilon}^{\Lambda} \frac{1}{y + x_0 + |y - x_0|} \frac{y u_0(y)}{y + u_0(y)^2} dy \\ &\geq \frac{\lambda}{2} \int_{\varepsilon}^{\Lambda} \frac{1}{y + x_0 + |y - x_0|} \frac{y t v_0(y)}{y + t^2 v_0(y)^2} dy. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{\lambda}{2} \int_{\varepsilon}^{\Lambda} \frac{1}{y + x_0 + |y - x_0|} \frac{y t v_0(y)}{y + t^2 v_0(y)^2} dy &> t \frac{\lambda}{2} \int_{\varepsilon}^{\Lambda} \frac{1}{y + x_0 + |y - x_0|} \frac{y v_0(y)}{y + v_0(y)^2} dy \\ &= t v_0(x_0), \end{aligned}$$

which contradicts $u_0(x_0) = t v_0(x_0)$ (see (3.1)). \square

Our proof of Theorem 2.2 is now complete.

4 Proof of Theorem 2.3

In this section we prove Theorem 2.3 in a sequence of lemmas.

An immediate consequence of the definition of the nonlinear integral operator B is the following.

Lemma 4.1. *Assume $0 < \lambda < 1$. Let W be as in (1.9). If $\psi \in W$, then $B\psi(x) \geq 0$ at all $x \in [\varepsilon, \infty)$.*

Lemma 4.2. *Assume $0 < \lambda < 1$. If $\psi \in W$, then $B\psi \in C[\varepsilon, \infty)$.*

Proof. Let $x_0 \in [\varepsilon, \infty)$ and let $x \in [\varepsilon, 2x_0]$. Then, for $\psi \in W$,

$$\begin{aligned} |B\psi(x) - B\psi(x_0)| &\leq \frac{\lambda}{2} \int_{\varepsilon}^{\infty} \left| \frac{\sqrt{x+\varepsilon}}{y+x+|y-x|} - \frac{\sqrt{x_0+\varepsilon}}{y+x_0+|y-x_0|} \right| \frac{\psi(y)}{\sqrt{y+\varepsilon}} dy \\ &= I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \frac{\lambda}{2} \int_{\varepsilon}^R \left| \frac{\sqrt{x+\varepsilon}}{y+x+|y-x|} - \frac{\sqrt{x_0+\varepsilon}}{y+x_0+|y-x_0|} \right| \frac{\psi(y)}{\sqrt{y+\varepsilon}} dy, \\ I_2 &= \frac{\lambda}{2} \int_R^{\infty} \left| \frac{\sqrt{x+\varepsilon}}{y+x+|y-x|} - \frac{\sqrt{x_0+\varepsilon}}{y+x_0+|y-x_0|} \right| \frac{\psi(y)}{\sqrt{y+\varepsilon}} dy. \end{aligned}$$

Let $\|\cdot\|_{B^0[\varepsilon, \infty)}$ denote the norm of the Banach space $B^0[\varepsilon, \infty)$. A straightforward calculation then gives for an arbitrary $\varepsilon_1 > 0$,

$$\begin{aligned} I_2 &\leq \lambda \|\psi\|_{B^0[\varepsilon, \infty)} \int_R^{\infty} \frac{\sqrt{2x_0+\varepsilon}}{y+\varepsilon} \frac{1}{\sqrt{y+\varepsilon}} dy \\ &< 2\lambda \|\psi\|_{B^0[\varepsilon, \infty)} \sqrt{\frac{2x_0+\varepsilon}{R}} \\ &< \frac{\varepsilon_1}{2}. \end{aligned}$$

Here, $R > \left(\frac{4\lambda \|\psi\|_{B^0[\varepsilon, \infty)} \sqrt{2x_0 + \varepsilon}}{\varepsilon_1} \right)^2$. We choose an R satisfying this inequality and fix it. Let us denote it by R_0 .

On the other hand, the function $(x, y) \mapsto \frac{\sqrt{x + \varepsilon}}{y + x + |y - x|}$ is uniformly continuous on $[\varepsilon, 2x_0] \times [\varepsilon, R_0]$, and hence there is a δ such that

$$\left| \frac{\sqrt{x + \varepsilon}}{y + x + |y - x|} - \frac{\sqrt{x_0 + \varepsilon}}{y + x_0 + |y - x_0|} \right| < \frac{\varepsilon_1}{2\lambda \|\psi\|_{B^0[\varepsilon, \infty)} \sqrt{R_0}}, \quad |x - x_0| < \delta.$$

Therefore, $I_1 < \varepsilon_1/2$. Thus $|B\psi(x) - B\psi(x_0)| < \varepsilon_1$, $|x - x_0| < \delta$. \square

Lemma 4.3. Assume $0 < \lambda < 1$. Let $\psi \in W$. Then $\|B\psi\|_{B^0[\varepsilon, \infty)} \leq \lambda \|\psi\|_{B^0[\varepsilon, \infty)}$. Consequently, $B\psi$ is bounded.

Proof.

$$B\psi(x) \leq \frac{\lambda}{2} \|\psi\|_{B^0[\varepsilon, \infty)} \sqrt{x + \varepsilon} \int_{\varepsilon}^{\infty} \frac{dy}{(y + x + |y - x|) \sqrt{y + \varepsilon}}.$$

Here, a straightforward calculation gives

$$\int_{\varepsilon}^{\infty} \frac{dy}{(y + x + |y - x|) \sqrt{y + \varepsilon}} \leq \frac{1}{\sqrt{x + \varepsilon} + \sqrt{\varepsilon}} + \frac{1}{\sqrt{\varepsilon}} \ln \left(\sqrt{1 + \frac{\varepsilon}{x}} + \sqrt{\frac{\varepsilon}{x}} \right).$$

Since

$$(4.1) \quad \sqrt{x + \varepsilon} \left\{ \frac{1}{\sqrt{x + \varepsilon} + \sqrt{\varepsilon}} + \frac{1}{\sqrt{\varepsilon}} \ln \left(\sqrt{1 + \frac{\varepsilon}{x}} + \sqrt{\frac{\varepsilon}{x}} \right) \right\} \leq 2 \quad (x \geq \varepsilon),$$

the result thus follows. \square

The three lemmas just above in this section imply the following.

Lemma 4.4. Assume $0 < \lambda < 1$. If $\psi \in W$, then $B\psi \in W$.

We now show that the nonlinear integral operator $B : W \rightarrow W$ is contractive. The following lemma is our key lemma in this section.

Lemma 4.5. Assume $0 < \lambda < 1$. Let $\psi, \varphi \in W$. Then

$$\|B\psi - B\varphi\|_{B^0[\varepsilon, \infty)} \leq \lambda \|\psi - \varphi\|_{B^0[\varepsilon, \infty)}.$$

Consequently, $B : W \rightarrow W$ is contractive.

Proof. Let $X, Y \geq 0$. A straightforward calculation gives

$$\begin{aligned} \left| \frac{yX}{y + X^2} - \frac{yY}{y + Y^2} \right| &= y|X - Y| \frac{|y - XY|}{(y + X^2)(y + Y^2)} \\ &\leq y|X - Y| \frac{y + 2XY}{y^2 + y(X^2 + Y^2)} \\ &\leq |X - Y|. \end{aligned}$$

Replacing X by $\psi(y)/\sqrt{y+\varepsilon}$ and Y by $\varphi(y)/\sqrt{y+\varepsilon}$ yields

$$\begin{aligned} |B\psi(x) - B\varphi(x)| &\leq \frac{\lambda}{2} \sqrt{x+\varepsilon} \int_{\varepsilon}^{\infty} \frac{1}{y+x+|y-x|} \left| \frac{\psi(y)}{\sqrt{y+\varepsilon}} - \frac{\varphi(y)}{\sqrt{y+\varepsilon}} \right| dy \\ &\leq \frac{\lambda}{2} \|\psi - \varphi\|_{B^0[\varepsilon, \infty)} \sqrt{x+\varepsilon} \int_{\varepsilon}^{\infty} \frac{dy}{(y+x+|y-x|) \sqrt{y+\varepsilon}} \\ &\leq \lambda \|\psi - \varphi\|_{B^0[\varepsilon, \infty)} \end{aligned}$$

by (4.1), from which the result follows. \square

The Banach fixed-point theorem (see e.g. Zeidler [18, p. 19]) thus implies the following.

Lemma 4.6. *Assume $0 < \lambda < 1$. Then the nonlinear integral operator $B : W \rightarrow W$ has a unique fixed point $\psi_0 \in W$.*

Clearly, the element $0 \in W$ is a fixed point of the operator $B : W \rightarrow W$ (see (1.10)). Moreover, the operator $B : W \rightarrow W$ has a unique fixed point $\psi_0 \in W$ by the lemma just above. Therefore, the fixed point $\psi_0 \in W$ is nothing but the element $0 \in W$. We thus have the following.

Lemma 4.7. *Assume $0 < \lambda < 1$. Then the nonlinear integral operator $B : W \rightarrow W$ has a unique fixed point $0 \in W$. Consequently, the massless abelian gluon model realizes the chiral symmetry.*

Our proof of Theorem 2.3 is now complete.

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